

Space complexity

Exercise sheet 11 (hints)

Exercise 11.1. Prove that if $A, B \in L$, then $A \cup B \in L$ and $A \cap B \in L$.

Hint: Reuse the work space of one decider for the other.

Exercise 11.2. Consider the alphabet $\{ (,) \}$ of *parentheses*. Prove that the language of all *balanced-parentheses* words is in L .

Hint: Keep track of the current balance as you scan the word.

Exercise 11.3. Let A , B , and C be languages. Prove the following statements about logspace reductions:

- (a) If $A \leq_L B$ and $B \leq_L C$, then $A \leq_L C$.
- (b) If $A \leq_L B$ and $B \in L$, then $A \in L$.

Hint: Recompute symbols of a reduction's output on demand.

A *nondeterministic* Turing machine may have several possible transitions for a state and observed symbol. To describe all its possible behaviours, imagine following every legal choice: whenever a configuration has several next moves, the computation branches, with one branch for each move. A path through this tree is a computation. The machine accepts an input if at least one computation halts in the accepting state, and rejects if all computations reject. It runs in polynomial time if every computation halts within a polynomial number of steps.

Exercise 11.4. Let NL be the class of languages decidable by a nondeterministic Turing machine in logarithmic space. Prove that the following problem DIRECTED-REACHABILITY belongs to NL .

Input: Directed graph G , nodes s and t .

Question: Is there a path from s to t ?

Hint: Guess a path nondeterministically, and bound its length.

Exercise 11.5. * Prove that the problem $\overline{2\text{COLOUR}}$ (non-2-colourability) belongs to NL.

Hint: A graph is not 2-colourable if and only if it contains an odd-length cycle.

Exercise 11.6. Prove that $L \in \text{NP}$ if and only if a polynomial-time nondeterministic Turing machine decides L .

Hint: Relate polynomial-length certificates to the choices along a computation.

Exercise 11.7. Let NEXP be the class of languages decided by exponential-time nondeterministic Turing machines. Prove that if $\text{P} = \text{NP}$, then $\text{EXP} = \text{NEXP}$.

Hint: Pad the input: $w \mapsto w \#\#\dots\#$. How many #'s do you need to add to link P to EXP?

Exercise 11.8. * Prove that $\text{P} \neq \text{SPACE}(n)$.

Hint: Combine the space hierarchy theorem with padding.