

Reductions and NP-completeness

Exercise sheet 9 & 10

Exercise 9.1. Recall that $L^* = \{w_1 \cdots w_r \mid r \geq 0, w_i \in L\}$. Suppose $L \in \text{NP}$. Is it true that $L^* \in \text{NP}$? Prove your answer.

Exercise 9.2. Prove that INDEPENDENT-SET is NP-complete, where

Input: A finite undirected graph G and a non-negative integer k encoded in binary.

Question: Does G contain a set of at least k vertices with no edge between any two of them?

Exercise 9.3. Prove that VERTEX-COVER is NP-complete, where

Input: A finite undirected graph G and a non-negative integer k encoded in binary.

Question: Does G have a set of at most k vertices that contains at least one endpoint of every edge?

Exercise 9.4. Prove that HITTING-SET is NP-complete, where

Input: A finite set U and a family $S_1, \dots, S_m \subseteq U$, with the elements and sets explicitly listed, and a non-negative integer k encoded in binary.

Question: Is there a set $H \subseteq U$ with $|H| \leq k$ such that $H \cap S_i \neq \emptyset$ for every $i \in \{1, \dots, m\}$?

Exercise 9.5. Prove that DOUBLE-SAT is NP-complete, where

Input: A Boolean formula φ , encoded as a word.

Question: Does φ have at least two distinct satisfying assignments?

Exercise 9.6. Prove that 3SAT is NP-complete, where

Input: A CNF formula with exactly three literal occurrences in every clause. A literal may occur more than once in a clause.

Question: Is the formula satisfiable?

Exercise 9.7. * Prove that SUBSET-SUM is NP-complete, where

Input: A list of non-negative integers $a_1, \dots, a_m$ and a non-negative integer t . All integers are encoded in binary.

Question: Is there an index set $I \subseteq \{1, \dots, m\}$ such that

$$\sum_{i \in I} a_i = t?$$

Exercise 9.8. * Prove that KNAPSACK is NP-complete, where

Input: A list of pairs $(w_1, v_1), \dots, (w_n, v_n)$ of non-negative integers, and non-negative integers W and V . All integers are encoded in binary.

Question: Is there an index set $I \subseteq \{1, \dots, n\}$ such that

$$\sum_{i \in I} w_i \leq W \quad \text{and} \quad \sum_{i \in I} v_i \geq V?$$

Exercise 9.9. * Prove that 3COLOUR is NP-complete, where

Input: A finite undirected graph G .

Question: Can the vertices of G be coloured with at most three colours so that the endpoints of every edge have different colours?

Exercise 9.10. * Prove that HAMILTONIAN-CYCLE is NP-complete, where

Input: A finite undirected graph G .

Question: Does G have a cycle that visits every vertex exactly once before returning to its starting vertex?