

# Reductions and NP-completeness

## Exercise sheet 9 & 10 (hints)

**Exercise 9.1.** Recall that  $L^* = \{w_1 \cdots w_r \mid r \geq 0, w_i \in L\}$ . Suppose  $L \in \text{NP}$ . Is it true that  $L^* \in \text{NP}$ ? Prove your answer.

*Hint: A certificate can record the cuts and a certificate for each factor.*

**Exercise 9.2.** Prove that INDEPENDENT-SET is NP-complete, where

**Input:** A finite undirected graph  $G$  and a non-negative integer  $k$  encoded in binary.

**Question:** Does  $G$  contain a set of at least  $k$  nodes with no edge between any two of them?

*Hint: What happens to a clique when edges and non-edges are exchanged?*

**Exercise 9.3.** Prove that VERTEX-COVER is NP-complete, where

**Input:** A finite undirected graph  $G$  and a non-negative integer  $k$  encoded in binary.

**Question:** Does  $G$  have a set of at most  $k$  nodes that contains at least one endpoint of every edge?

*Hint: If a set covers every edge, what can you say about the nodes outside it?*

**Exercise 9.4.** Prove that HITTING-SET is NP-complete, where

**Input:** A finite set  $U$  and a family  $S_1, \dots, S_m \subseteq U$ , with the elements and sets explicitly listed, and a non-negative integer  $k$  encoded in binary.

**Question:** Is there a set  $H \subseteq U$  with  $|H| \leq k$  such that  $H \cap S_i \neq \emptyset$  for every  $i \in \{1, \dots, m\}$ ?

*Hint: Reduce from VERTEX-COVER.*

**Exercise 9.5.** Prove that DOUBLE-SAT is NP-complete, where

**Input:** A Boolean formula  $\varphi$ , encoded as a word.

**Question:** Does  $\varphi$  have at least two distinct satisfying assignments?

*Hint: Can you double the number of possible assignments?*

**Exercise 9.6.** Prove that 3SAT is NP-complete, where

**Input:** A CNF formula with exactly three literal occurrences in every clause. A literal may occur more than once in a clause.

**Question:** Is the formula satisfiable?

*Hint: Start with a clause of four literals. Split it into two clauses using a fresh variable, once positively and once negatively. Extend the idea to longer clauses.*

**Exercise 9.7.** \* Prove that SUBSET-SUM is NP-complete, where

**Input:** A list of non-negative integers  $a_1, \dots, a_m$  and a non-negative integer  $t$ . All integers are encoded in binary.

**Question:** Is there an index set  $I \subseteq \{1, \dots, m\}$  such that

$$\sum_{i \in I} a_i = t?$$

*Hint: Reduce from 3SAT. Let  $\varphi$  have variables  $x_1, \dots, x_n$  and clauses  $C_1, \dots, C_m$ . Work with numbers that have  $n + m$  decimal digits.*

**Exercise 9.8.** \* Prove that KNAPSACK is NP-complete, where

**Input:** A list of pairs  $(w_1, v_1), \dots, (w_n, v_n)$  of non-negative integers, and non-negative integers  $W$  and  $V$ . All integers are encoded in binary.

**Question:** Is there an index set  $I \subseteq \{1, \dots, n\}$  such that

$$\sum_{i \in I} w_i \leq W \quad \text{and} \quad \sum_{i \in I} v_i \geq V?$$

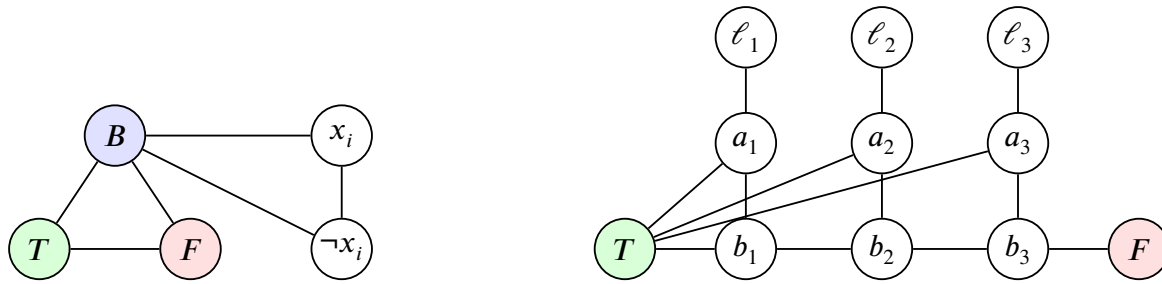
*Hint: Reduce from SUBSET-SUM.*

**Exercise 9.9.** \* Prove that 3COLOUR is NP-complete, where

**Input:** A finite undirected graph  $G$ .

**Question:** Can the nodes of  $G$  be coloured with at most three colours so that the endpoints of every edge have different colours?

*Hint: Reduce from 3SAT. The gadget on the left sets up the three colours  $T, F, B$  and encodes each variable using the nodes  $x_i$  and  $\neg x_i$ . What colours can this pair receive?*



The gadget on the right is for a clause with literal nodes  $\ell_1, \ell_2, \ell_3$  and six fresh nodes  $a_i, b_i$ . Its  $T$  and  $F$  nodes are the same palette nodes as on the left. Suppose each literal node is already in either  $T$  or  $F$ . What does the gadget on the right enforce?

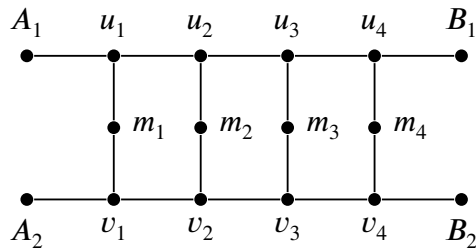
**Exercise 9.10.** \* Prove that HAMILTONIAN-CYCLE is NP-complete, where

**Input:** A finite undirected graph  $G$ .

**Question:** Does  $G$  have a cycle that visits every node exactly once before returning to its starting node?

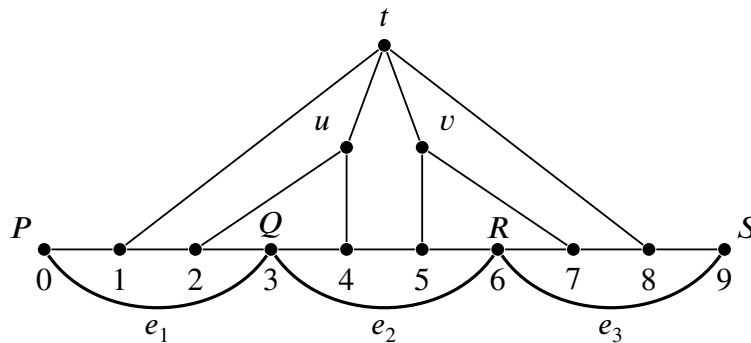
*Hint: Reduce from 3SAT. Here are two gadgets.*

The first has twelve internal nodes;  $A_1, B_1, A_2, B_2$  are outside nodes. No other connections to the outside are allowed.



Suppose this is a part of a bigger graph, connected to it via the external nodes  $A_i$  and  $B_i$ . What are the ways the Hamiltonian cycle could traverse it?

The second gadget has three distinguished thick edges  $e_1, e_2, e_3$ .



Look for a path from  $P = 0$  to  $S = 9$  visiting all thirteen nodes. Which subsets of the three thick edges can such a path use?