

Midterm Demo

Five problems, 100 primary points Duration: 90 minutes

Each problem is worth 20 points, with subexercises of equal weight. Partial credit is available for correct intermediate steps. The mark out of 20 is:

$$\left\lceil \frac{1}{5} \times \text{the number of points earned} \right\rceil$$

Problem 1 (Decidability and enumerability). Use a fixed effective encoding of Turing machines. For each language below, state whether it is decidable and whether it is enumerable, and justify your answer.

(a)

$$\text{EVEN} = \{w \in \{0, 1\}^* \mid w \text{ represents an even number}\}$$

(5 points)

(b)

$$\text{CPP-HALT-ON-EMPTY} = \{\pi \text{ program in C++} \mid \pi \text{ halts on input } \varepsilon\}$$

(5 points)

(c)

$$\text{DIRECTED-ACYCLIC} = \{G \text{ directed graph} \mid G \text{ has no cycles}\}$$

(5 points)

(d)

EQUIVALENT-PROGRAMS = $\{\langle \pi, \xi \rangle \text{programs in Python} \mid \text{on every input, } \pi \text{ and } \xi$
return the same output or are both undefined}

(5 points)

Problem 2 (Turing machines). For each of the following modifications of Turing machines, tell if it reduces, increases, or preserves the computational power (i.e., the class of functions that such machines can compute; running time is irrelevant).

(a) One-way infinite tape model, with cells indexed by $0, 1, 2, \dots$ and a left move at cell 0 leaving the head there.

(5 points)

(b) A machine with five tapes, each with its own read/write head.

(5 points)

(c) A machine that has a built-in instant SAT-solver: a special tape, on which it can write a Boolean formula, and at the next moment it will receive a satisfying assignment for that formula or the answer that the formula is unsatisfiable.

(5 points)

(d) A machine that cannot move outside of the area where the input is written, but which can overwrite cells in that area, as usual.

(5 points)

Problem 3 (Lambda calculus). Recall the Church numerals and the encodings of Booleans and pairs:

$$\bar{n} = \lambda f. \lambda x. f^n(x), \quad \text{TRUE} = \lambda x. \lambda y. x, \quad \text{FALSE} = \lambda x. \lambda y. y,$$

$$\text{PAIR} = \lambda a. \lambda b. \lambda h. h a b, \quad \text{FST} = \lambda p. p \text{ TRUE}, \quad \text{SND} = \lambda p. p \text{ FALSE}.$$

Write lambda terms for the following functions on Church numerals. You may use definitions from earlier parts. Define any other helper terms that you introduce.

(a) Plus2: $\bar{n} \mapsto \overline{n+2}$. (5 points)

(b) Test > 0: $\bar{0} \mapsto \text{FALSE}$ and $\bar{n} \mapsto \text{TRUE}$ for $n > 0$. (5 points)

(c) Addition: $\bar{m}, \bar{n} \mapsto \overline{m+n}$. (5 points)

(d) Compare the pairs: $(\bar{m}_1, \bar{n}_1) < (\bar{m}_2, \bar{n}_2)$ iff $m_1 < m_2$, or $m_1 = m_2$ and $n_1 < n_2$.
So $(\bar{m}_1, \bar{n}_1), (\bar{m}_2, \bar{n}_2) \mapsto \text{TRUE}$ if $(\bar{m}_1, \bar{n}_1) < (\bar{m}_2, \bar{n}_2)$ and FALSE otherwise. (5 points)

Problem 4 (Upper bounds).

(a) Prove that 5COLOUR, the problem of deciding whether a graph has a proper colouring with five colours, belongs to NP. (5 points)

(b) Prove that $\text{PRIMES} = \{w \in \{0, 1\}^* \mid w \text{ represents a prime number}\}$ belongs to NP. (5 points)

(c) Prove that $\text{UNARY-PRIMES} = \{w \in \{1\}^* \mid |w| \text{ is prime}\}$ belongs to L. (5 points)

(d) Prove that 2COLOUR, the problem of deciding whether a graph has a proper colouring with two colours, belongs to P. (5 points)

Problem 5 (Lower bounds). (a) Prove that the following INDEPENDENT-SET problem is NP-complete:

Input: Graph G , number k .

Question: Does G contain a set of k nodes, no two of which form an edge?

Hint: reduce from CLIQUE.

(10 points)

(b) Prove that the following HITTING-SET problem is NP-complete:

Input: A finite set U , a family of subsets $S_1, \dots, S_m \subseteq U$, and a nonnegative integer k .

Question: Is there a set $H \subseteq U$ with $|H| \leq k$ such that $H \cap S_i \neq \emptyset$ for every i ?

(10 points)